Periodic Orbits on a Smooth Surface.

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§ 1. *Introduction.*

In the third chapter of Moulton's "Periodic Orbits," which will appear shortly, the general solutions of the problem of the spherical pendulum are determined as power series in a parameter in which the coefficients are transcendental functions of the time. The solution obtained for the vertical motion is periodic, but the solutions for the horizontal are not periodic in general. The problem discussed in this paper is a generalization of the problem of the spherical pendulum. The path described by the bob of the pendulum may be considered as the orbit described by a particle which moves, subject to gravity, on the surface of a smooth sphere whose radius is equal to the length of the pendulum and whose center is at the point of suspension. The object of this paper is to show the existence and to give a method for the determination of periodic orbits described by a particle which moves under similar conditions on a smooth surface of more general character. The orbits determined have the same period as the solution for the vertical motion of the spherical pendulum. The equation of the surface is the same as that used by Poincaré in his memoir "Sur les Lignes Géodésiques des Surfaces Convexes," *Trans. Am. Math. Soc.*, Vol. VI (July, 1905). Periodic orbits play the same rôle in the present paper as closed geodesic lines in Poincaré's memoir.

§ 2. *The Differential Equations.*

Let us take a system of rectangular axes with the positive z-axis directed upwards. We shall denote the surface upon which the particle is constrained to move by the equation

$$F(x, y, z) \equiv x^2 + y^2 + z^2 - l^2 + 2\,\varepsilon f(x, y, z) = 0, \tag{1}$$

where ε is a parameter and $f(x, y, z)$ is a power series of the form

$$f(x, y, z) = \sum_{i,j,k=0}^{\infty} f_{ijk}\, x^i\, y^j\, z^k, \tag{2}$$

the f_{ijk} being constants, and it converges for $|x|$, $|y|$ and $|z|$ sufficiently small.

If (2) is a polynomial, then no restrictions need be placed on x, y and z, except that they shall be finite. For $\varepsilon = 0$ the equation (1) reduces to the equation of a sphere with center at the origin and radius l.

Let us choose the unit of mass so that the mass of the particle is unity. Then, denoting derivatives with respect to the time by accents, we obtain as the differential equations of motion for the particle,

$$x'' = X, \quad y'' = Y, \quad z'' = Z - g, \tag{3}$$

where X, Y, Z are the normal reactions due to the surface. Since the surface is assumed to be smooth, the normal reactions at any point are proportional to the direction cosines of the normal at that point. Hence,

$$\left.\begin{aligned} X &= \lambda F_x = 2\lambda\,[x + \varepsilon f_x], \\ Y &= \lambda F_y = 2\lambda\,[y + \varepsilon f_y], \\ Z &= \lambda F_z = 2\lambda\,[z + \varepsilon f_z], \end{aligned}\right\} \tag{4}$$

where λ is a factor of proportionality. When equations (4) are substituted in (3), the differential equations become

$$x'' = 2\lambda\,[x + \varepsilon f_x], \quad y'' = 2\lambda\,[y + \varepsilon f_y], \quad z'' = 2\lambda\,[z + \varepsilon f_z] - g. \tag{5}$$

These equations admit the integral

$$x'^2 + y'^2 + z'^2 = g\,(-2z + c_1), \tag{6}$$

where c_1 is the constant of integration.

In order to determine λ, we find the second derivative of the equation of constraint with respect to t and eliminate x'', y'', z'', and $x'^2 + y'^2 + z'^2$ by (5) and (6). Since $F(x, y, z)$ is independent of t, we have $F'' = 0$, and therefore, when the differentiations and eliminations are made, we obtain

$$2\lambda = \frac{g\,(3z - c_1) - \varepsilon\,[x'^2 f_{xx} + y'^2 f_{yy} + z'^2 f_{zz} + 2y'z' f_{yz} + 2x'z' f_{xz} + 2x'y' f_{xy} - g f_z]}{l^2 + 2\varepsilon\,[x f_x + y f_y + z f_z - f] + \varepsilon^2\,[f_x^2 + f_y^2 + f_z^2]}. \tag{7}$$

After the values of the partial derivatives are obtained from (2) and substituted in (7), the expression for 2λ can be arranged as a power series of the form

$$2\lambda = \frac{g}{l^2}\,(3z - c_1) + \lambda_1 \varepsilon + \lambda_2 \varepsilon^2 + \dots, \tag{8}$$

in which the coefficients λ_j are power series in x, y, z and contain x', y', z' to the second degree considered together. If $f(x, y, z)$ is a polynomial, then the convergence of (8) can be controlled by ε alone, provided x, y and z are finite. If $f(x, y, z)$ is a power series, then x, y and z must lie in the region for which (2) converges. Upon substituting (8) in (5), the differential equations take the form

$$\left.\begin{aligned} x'' &= \frac{g}{l^2}(3z - c_1)\,x + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots, \\ y'' &= \frac{g}{l^2}(3z - c_1)\,y + \varepsilon Y_1 + \varepsilon^2 Y_2 + \ldots, \\ z'' &= \frac{g}{l^2}(3z - c_1)\,z - g + \varepsilon Z_1 + \varepsilon^2 Z_2 + \ldots, \end{aligned}\right\} \tag{9}$$

where X_j, Y_j, Z_j are similar in form to the λ_j in (8).

§ 3. *The Spherical Pendulum.*

For $\varepsilon = 0$ the equations (9) reduce to the differential equations of the spherical pendulum, which are

$$x'' = \frac{g}{l^2}(3z - c_1)\,x, \quad y'' = \frac{g}{l^2}(3z - c_1)\,y, \quad z'' = \frac{g}{l^2}(3z - c_1)\,z - g. \tag{10}$$

The last equation of (10) is independent of the other two and is solved first. It admits the integral

$$\begin{aligned} z'^2 &= \frac{g}{l^2}(2z - c_1)\,z^2 - g\,(2z - c_2) \\ &= \frac{2g}{l^2}(z - \alpha_1)(z - \alpha_2)(z - \alpha_3), \qquad (\alpha_1 \geqq \alpha_2 \geqq \alpha_3), \end{aligned} \tag{11}$$

where c_2 is the constant of integration. The periodic solution of the last equation of (10) has been obtained in Professor Moulton's memoir, to which reference has already been made. The solution is

$$z = \psi(\tau) = \alpha_3 + (\alpha_1 - \alpha_3)\left[\frac{1}{2}(1 - \cos 2\tau)\,\mu + \frac{1}{32}(1 - \cos 4\tau)\,\mu^2 + \ldots\right], \tag{12}$$

where

$$\left.\begin{aligned} \tau &= \sqrt{\frac{g(\alpha_1 - \alpha_3)}{2l^2(1+\delta)}}\,(t - t_0), \\ \delta &= \frac{1}{2}\mu + \frac{11}{32}\mu^2 + \ldots, \\ \mu &= \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}, \qquad (0 \leqq \mu \leqq 1). \end{aligned}\right\} \tag{13}$$

In the physical problem of the spherical pendulum, excluding the case of revolution in the xy-plane with infinite speed and that of the simple pendulum, the three constants α_1, α_2, α_3 satisfy the inequalities

$$-l < \alpha_3 < 0, \quad -l < \alpha_2 < +l, \quad \alpha_1 > +l.$$

On comparing the equations in (11), it is seen that

$$2(\alpha_1 + \alpha_2 + \alpha_3) = c_1, \quad \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = -l^2, \quad 2\alpha_1\alpha_2\alpha_3 = -c_2 l^2. \tag{14}$$

§ 4. *The Solutions of the Equations of Variation.*

Let
$$z = \psi + w, \tag{15}$$
where ψ is the periodic function defined in (12). The function w is undetermined except that it vanishes with ε. Now let (15) and (13) be substituted in (9). If derivatives with respect to τ are denoted by dots, then the differential equations (9) become

$$\left.\begin{aligned}
\ddot{x} + [a^2 + \theta_1^{(1)}\mu + \theta_2^{(1)}\mu^2 + \dots]\,x &= \frac{6\,(1+\delta)\,x\,w}{\alpha_1 - \alpha_3} + \varepsilon\,\bar{X}_1 + \varepsilon^2\,\bar{X}_2 + \dots,\\
\ddot{y} + [a^2 + \theta_1^{(1)}\mu + \theta_2^{(1)}\mu^2 + \dots]\,y &= \frac{6\,(1+\delta)\,y\,w}{\alpha_1 - \alpha_3} + \varepsilon\,\bar{Y}_1 + \varepsilon^2\,\bar{Y}_2 + \dots,\\
\ddot{w} + [b^2 + \theta_1^{(2)}\mu + \theta_2^{(2)}\mu^2 + \dots]\,w &= \frac{6\,(1+\delta)\,w^2}{\alpha_1 - \alpha_3} + \varepsilon\,W_1 + \varepsilon^2\,W_2 + \dots,
\end{aligned}\right\} \tag{16}$$

where a^2 and b^2 are constants independent of μ; and $\theta_j^{(i)}\,(i=1,2;\ j=1,\dots,\infty)$ are sums of cosines of even multiples of τ and are therefore periodic with the period 2π. The $\bar{X}_j$, $\bar{Y}_j$, W_j appearing in the right members of (16) are power series in x, y, w in which the coefficients are similar in form to $\psi(\tau)$ except that they contain additional terms in $\dot{x}$, $\dot{y}$, $\dot{w}$, $\dot{\psi}$ to the second degree considered together.

By putting $\varepsilon = 0$ in (16), and taking only the linear terms in x, y, and w, we obtain the equations of variation, which are

$$\left.\begin{aligned}
\ddot{x} + [a^2 + \theta_1^{(1)}\mu + \theta_2^{(1)}\mu^2 + \dots]\,x &= 0,\\
\ddot{y} + [a^2 + \theta_1^{(1)}\mu + \theta_2^{(1)}\mu^2 + \dots]\,y &= 0,\\
\ddot{w} + [b^2 + \theta_1^{(2)}\mu + \theta_2^{(2)}\mu^2 + \dots]\,w &= 0.
\end{aligned}\right\} \tag{17}$$

Obviously the solutions of the first two equations of (17) can differ only in the arbitrary constants. The solutions of these two equations have been obtained in Professor Moulton's memoir, where it is shown that there are three forms of the solutions according to the values of a:

CASE I. $a \neq 0$ and $2a$ not an integer.
CASE II. $a \neq 0$ and $2a$ an integer.
CASE III. $a = 0$.

CASE I. This is considered as the general case. The solutions in this case have the form

$$\left.\begin{aligned}
x &= A_1\,e^{\alpha\sqrt{-1}\,\tau}\,\xi_1 + A_2\,e^{-\alpha\sqrt{-1}\,\tau}\,\xi_2,\\
y &= A_3\,e^{\alpha\sqrt{-1}\,\tau}\,\xi_1 + A_4\,e^{-\alpha\sqrt{-1}\,\tau}\,\xi_2,
\end{aligned}\right\} \tag{18}$$

where $A_i\ (i=1,\dots,4)$ are the constants of integration; α is a power series

in μ with constant coefficients; and ξ_1, ξ_2 are power series in μ in which the coefficients have the form

$$\xi_1 = \sum_j [a_j \cos 2 j \tau + \sqrt{-1}\, b_j \sin 2 j \tau],$$
$$\xi_2 = \sum_j [a_j \cos 2 j \tau - \sqrt{-1}\, b_j \sin 2 j \tau],$$

a_j and b_j denoting real constants. These solutions have the additional property

$$\xi_1(0) = \xi_2(0) = 1.$$

CASE II. In this case the solutions are similar in form to (18), but they contain, in addition, terms in $\cos 2\, a\, j \tau$ and $\sqrt{-1} \sin 2\, a\, j \tau$. The constants of integration are determined so that

$$\dot{\xi}_1(0) = \dot{\xi}_2(0) = 1.$$

CASE III. For $a = 0$ the solutions have the same form as (18) except that α, ξ_1, ξ_2 are power series in $\sqrt{\mu}$ instead of μ.

Unless otherwise stated we shall suppose that we are dealing with Case I.

We shall now derive the solutions of the last equation of (17). Since the differential equations (9) do not contain t explicitly, it is known, *à priori*,* that two of the characteristic exponents which belong to the solutions of equations (17) are zero. These two zero exponents belong to the solution of the last equation of (17). The generating solution of this equation is $\psi(\tau)$, defined in (12), and it contains two arbitrary constants, viz., the initial time t_0, and the scale factor l, the latter occurring only in the second degree. Hence, the two solutions of the last equation of (17) are†

$$\begin{aligned} w_1 &= \frac{\partial}{\partial t_0} \psi \left\{ \sqrt{\frac{g(\alpha_1 - \alpha_3)}{2 l^2 (1+\delta)}}\, (t - t_0) \right\} = -\frac{\partial}{\partial \tau} \psi(\tau) \\ &= \mu (\alpha_3 - \alpha_1) \left[\sin 2\tau + \frac{1}{8} \mu \sin 4\tau + \ldots \right], \\ w_2 &= \frac{\partial \psi}{\partial l^2}. \end{aligned}$$

Since each of these solutions is multiplied later by an arbitrary constant, the factor $\mu(\alpha_3 - \alpha_1)$ may be absorbed by the arbitrary constant, and the first solution becomes

$$\bar{w}_1 = \phi = \sin 2\tau + \frac{1}{8} \mu \sin 4\tau + \ldots . \tag{19}$$

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* Moulton, "Periodic Orbits," § 33. Poincaré, "Les Méthodes Nouvelles de la Mécanique Céleste," Vol. I, Chap. 4.

† Moulton, *loc. cit.*, § 32. Poincaré, *loc. cit.*, Vol. I, Chap. 4.

It is observed from equations (12) and (13) that ψ is a function of α_1, α_3, μ and τ, while τ is a function of α_1, α_3, δ and l^2, and δ is a function of μ. Now when α_2 is eliminated from (14) by the substitution

$$\alpha_2 = \alpha_3 + \mu\,(\alpha_1 - \alpha_3),$$

the first equation of (14) expresses μ as a function of α_1 and α_3, and the other two equations express α_1 and α_3 as functions of l^2. The constants c_1 and c_2 enter into these relations, but as they are constants of integration, they are independent of the scale factor. Hence,

$$\begin{aligned}\frac{\partial\psi}{\partial l^2} &= \left(\frac{\partial\psi}{\partial\alpha_1}\right)\frac{\partial\alpha_1}{\partial l^2} + \left(\frac{\partial\psi}{\partial\alpha_3}\right)\frac{\partial\alpha_3}{\partial l^2} + \left(\frac{\partial\psi}{\partial\mu}\right)\left[\frac{\partial\mu}{\partial\alpha_1}\frac{\partial\alpha_1}{\partial l^2} + \frac{\partial\mu}{\partial\alpha_3}\frac{\partial\alpha_3}{\partial l^2}\right] \\ &+ \frac{\partial\psi}{\partial\tau}\left[\left(\frac{\partial\tau}{\partial l^2}\right) + \frac{\partial\tau}{\partial\delta}\frac{\partial\delta}{\partial\mu}\left\{\frac{\partial\mu}{\partial\alpha_1}\frac{\partial\alpha_1}{\partial l^2} + \frac{\partial\mu}{\partial\alpha_3}\frac{\partial\alpha_3}{\partial l^2}\right\} + \left(\frac{\partial\tau}{\partial\alpha_1}\right)\frac{\partial\alpha_1}{\partial l^2} + \left(\frac{\partial\tau}{\partial\alpha_3}\right)\frac{\partial\alpha_3}{\partial l^2}\right],\end{aligned} \tag{20}$$

where the parentheses ( ) denote that the differentiation is performed only in so far as the variable enters the function explicitly. Upon performing the differentiations expressed in (20), we obtain

$$\begin{aligned}&\left(\frac{\partial\psi}{\partial\alpha_1}\right) = 1 - \left(\frac{\partial\psi}{\partial\alpha_3}\right) = \frac{1}{2}(1-\cos 2\tau)\,\mu + \frac{1}{32}(1-\cos 4\tau)\,\mu^2 + \ldots, \\ &\frac{\partial\alpha_1}{\partial l^2} = \frac{1}{(\alpha_1-\alpha_3)(1-\mu)}, \quad \frac{\partial\alpha_3}{\partial l^2} = \frac{1}{\mu\,(\alpha_3-\alpha_1)} = \frac{2}{2\,\alpha_1 + 4\,\alpha_3 - c_1}, \\ &\frac{\partial\mu}{\partial\alpha_1} = \frac{1+\mu}{\alpha_3-\alpha_1}, \quad \frac{\partial\mu}{\partial\alpha_3} = \frac{2-\mu}{\alpha_3-\alpha_1}, \quad \frac{\partial\psi}{\partial\tau} = \mu\,(\alpha_1-\alpha_3)\,\phi, \\ &\left(\frac{\partial\psi}{\partial\mu}\right) = (\alpha_1-\alpha_3)\left[\frac{1}{2}(1-\cos 2\tau) + \frac{1}{16}(1-\cos 4\tau)\,\mu + \ldots\right], \\ &\frac{\partial\tau}{\partial\delta} = -\frac{\tau}{2\,(1+\delta)}, \quad \frac{\partial\delta}{\partial\mu} = \frac{1}{2} + \frac{11}{16}\mu + \ldots, \\ &\left(\frac{\partial\tau}{\partial l^2}\right) = -\frac{\tau}{2\,l^2}, \quad \left(\frac{\partial\tau}{\partial\alpha_1}\right) = -\left(\frac{\partial\tau}{\partial\alpha_3}\right) = \frac{\tau}{2\,(\alpha_1-\alpha_3)}.\end{aligned}$$

When these results are substituted in (20), we get for the second solution

$$w_2 = \chi + A\,\tau\,\phi, \tag{21}$$

where χ and A are power series in μ of the form

$$\begin{aligned}\chi = &\frac{1}{2(\alpha_3-\alpha_1)\,(2\alpha_1+4\,\alpha_3-c_1)}\,[2\,\alpha_1 + 4\,\alpha_3 - c_1 - (6\,\alpha_1 - c_1)\cos 2\tau \\ &+ \frac{\mu}{8}\{22\alpha_1 + 32\alpha_3 - 9\,c_1 - 8\,(2\,\alpha_1 + 4\,\alpha_3 - c_1)\cos 2\tau - (6\,\alpha_1 - c_1)\cos 4\tau\} + \ldots], \\ A = &\frac{8\,(\alpha_3-\alpha_1)^2 - 9\,l^2}{16\,l^2\,(\alpha_3-\alpha_1)}\,\mu + \ldots.\end{aligned}$$

The determinant of the two solutions (19) and (21) at $\tau = 0$ is

$$D = -\dot{\phi}(0)\,\chi(0) = -\frac{4}{2\,\alpha_1 + 4\,\alpha_3 - c_1} + \text{terms in } \mu, \tag{22}$$

which is not zero for μ sufficiently small unless α_1, α_3 or c_1 is infinite. Now α_3 is finite because of the inequality $-l < \alpha_3 < 0$. It is shown in Professor Moulton's memoir that if c_1 is infinite, the particle revolves in the xy-plane with infinite speed, and, of course, this case can not be realized physically. Then c_1 is finite, and since $-l < \alpha_3 < 0$, $-l < \alpha_2 < +l$, it follows from the first equation of (14) that α_1 also is finite. Hence, $D \neq 0$ for μ sufficiently small, and the two solutions (19) and (21) constitute a fundamental set. The general solution of the last equation of (17) is therefore

$$w = A_5 \phi + A_6 [\chi + A\,\tau\,\phi], \tag{23}$$

where A_5, A_6 are the arbitrary constants. The functions ϕ and χ are periodic in τ with the period 2π.

§ 5. *Existence of Periodic Solutions.*

Let us take the initial conditions at $\tau = 0$,

$$x = a_1, \quad x = a_2, \quad y = a_3, \quad y = a_4, \quad w = a_5, \quad w = a_6. \tag{24}$$

If the surface of constraint is a closed surface, then the vertical velocity of the particle must vanish for some value of t and change sign. If the surface is not closed and if the motion of the particle is to be periodic, then x', y', z' must vanish and change signs for some values of t, otherwise the particle would recede to infinity. Then without loss of generality t_0 can be chosen as the time when the vertical velocity is zero. Hence, $z = 0$ at $\tau = 0$; and since $\dot{\psi}(0) = 0$, it follows from (15) that $\dot{w}(0) = 0$. Therefore, a_6 in (24) may be put equal to zero.

In order to prove the existence of periodic solutions of equations (16), we integrate (16) as power series in $a_i\,(i = 1, \ldots, 5)$ and ε, and impose necessary conditions that x, y and w shall be periodic in τ with the period 2π. Only the linear terms in a_i are required in explicit form. The solutions of (16), subject to the initial conditions (24), are

$$\left.\begin{aligned}
x &= \left(\frac{a_1}{2} - \frac{a_2}{\Delta}\right) e^{\alpha\sqrt{-1}\,\tau}\,\xi_1 + \left(\frac{a_1}{2} + \frac{a_2}{\Delta}\right) e^{-\alpha\sqrt{-1}\,\tau}\,\xi_2 + \varepsilon P_1(a_i, \varepsilon;\ \tau),\\
y &= \left(\frac{a_3}{2} - \frac{a_4}{\Delta}\right) e^{\alpha\sqrt{-1}\,\tau}\,\xi_1 + \left(\frac{a_3}{2} + \frac{a_4}{\Delta}\right) e^{-\alpha\sqrt{-1}\,\tau}\,\xi_2 + \varepsilon P_2(a_i, \varepsilon;\ \tau),\\
w &= \frac{a_5}{\chi(0)}\,[\chi + A\,\tau\,\phi] + \varepsilon P_3(a_i, \varepsilon;\ \tau),
\end{aligned}\right\} \tag{25}$$

where Δ is the determinant of the fundamental set of solutions (18) and is therefore different from zero. The terms P_1, P_2 and P_3 are power series in a_i and ε, and carry ε as a factor since the right members of (16) vanish with ε.

Sufficient conditions that x, y, w in (25) shall be periodic in τ with the period 2π are

$$\left.\begin{array}{lll} x(2\pi)-x(0)=0, & y(2\pi)-y(0)=0, & w(2\pi)-w(0)=0, \\ \dot{x}(2\pi)-\dot{x}(0)=0, & \dot{y}(2\pi)-\dot{y}(0)=0, & \dot{w}(2\pi)-\dot{w}(0)=0. \end{array}\right\} \quad (26)$$

These six conditions are not independent, as we shall show, and the condition $w(2\pi)-w(0)=0$ is a consequence of the other five conditions. In order to show this we make use of the integral (6), which, on being transformed by the substitutions (13) and (15), takes the form

$$\dot{x}^2+\dot{y}^2+(\dot{\psi}+\dot{w})^2+\frac{2\,l^2(1+\delta)}{\alpha_1-\alpha_3}(2\,\psi+2\,w-c_1)=0. \quad (27)$$

Let us make in (27) the usual substitutions

$$\left.\begin{array}{lll} x=x(0)+\bar{x}, & y=y(0)+\bar{y}, & w=w(0)+\bar{w}, \\ \dot{x}=\dot{x}(0)+\dot{\bar{x}}, & \dot{y}=\dot{y}(0)+\dot{\bar{y}}, & \dot{w}=0+\dot{\bar{w}}, \end{array}\right\} \quad (28)$$

where $\bar{x}, \ldots, \dot{\bar{w}}$ vanish at $\tau=0$; and let the resulting equations be denoted by (27a). By putting $\tau=0$ in (27a) we obtain an equation (27b) connecting the constant terms of (27a) which are independent of $\bar{x}, \ldots, \dot{\bar{w}}$. When those constant terms are eliminated from (27a) by means of (27b), there results an equation of the form

$$G(\bar{x}, \bar{y}, \bar{w}, \dot{\bar{x}}, \dot{\bar{y}}, \dot{\bar{w}})=0, \quad (29)$$

in which, at $\tau=2\pi$, there are no terms independent of the arguments indicated. The coefficient of the linear term in $w(2\pi)$ is $4\,l^2(1+\delta)/(\alpha_1-\alpha_3)$, and it is different from zero. Hence, by the theory of implicit functions, equation (29) can be solved uniquely for $\bar{w}(2\pi)$ as a power series of the form

$$\bar{w}(2\pi)=Q\{\bar{x}(2\pi),\ \bar{y}(2\pi),\ \dot{\bar{x}}(2\pi),\ \dot{\bar{y}}(2\pi),\ \dot{\bar{w}}(2\pi)\}, \quad (30)$$

in which there is no constant term. Now if the conditions in (26), except the condition $w(2\pi)-w(0)=0$, are imposed, then

$$\bar{x}(2\pi)=\bar{y}(2\pi)=\dot{\bar{x}}(2\pi)=\dot{\bar{y}}(2\pi)=\dot{\bar{w}}(2\pi)=0, \quad (31)$$

and it follows from (30) and (31) that $\bar{w}(2\pi)=0$. Therefore, the condition $w(2\pi)-w(0)=0$ is a consequence of the other five conditions in (26) and may be suppressed.

When the necessary conditions of (26) are imposed upon the solutions (25), the equations which a_i $(i=1, \ldots, 5)$ must satisfy are found to be

$$\left.\begin{aligned}
0 &= \frac{a_1}{2}\left[e^{2a\sqrt{-1}\pi} + e^{-2a\sqrt{-1}\pi} - 2\right] - \frac{a_2}{\Delta}\left[e^{2a\sqrt{-1}\pi} - e^{-2a\sqrt{-1}\pi}\right] + \text{terms in } \varepsilon,\\
0 &= -\frac{\Delta a_1}{4}\left[e^{2a\sqrt{-1}\pi} - e^{-2a\sqrt{-1}\pi}\right] + \frac{a_2}{2}\left[e^{2a\sqrt{-1}\pi} + e^{-2a\sqrt{-1}\pi} - 2\right] + \text{terms in } \varepsilon,\\
0 &= \frac{a_3}{2}\left[e^{2a\sqrt{-1}\pi} + e^{-2a\sqrt{-1}\pi} - 2\right] - \frac{a_4}{\Delta}\left[e^{2a\sqrt{-1}\pi} - e^{-2a\sqrt{-1}\pi}\right] + \text{terms in } \varepsilon,\\
0 &= -\frac{\Delta a_3}{4}\left[e^{2a\sqrt{-1}\pi} - e^{-2a\sqrt{-1}\pi}\right] + \frac{a_4}{2}\left[e^{2a\sqrt{-1}\pi} + e^{-2a\sqrt{-1}\pi} - 2\right] + \text{terms in } \varepsilon,\\
0 &= \frac{2\pi a_5 A \dot{\phi}(2\pi)}{\chi(0)} + \text{terms in } \varepsilon.
\end{aligned}\right\} \quad (32)$$

The determinant of the linear terms in a_i $(i = 1, \ldots, 5)$ is

$$2\pi A \frac{\dot{\phi}(2\pi)}{\chi(0)} (e^{2a\sqrt{-1}\pi} - 1)^2 (e^{-2a\sqrt{-1}\pi} - 1)^2, \quad (33)$$

and it is different from zero if α is not a real integer or zero. First, let us suppose that α is not an integer. Then the determinant (33) is not zero and the equations (32) can be solved uniquely for a_i as power series in ε. These series vanish with ε and converge for $|\varepsilon|$ sufficiently small. Hence, periodic solutions of (16) exist uniquely and have the form

$$x = \sum_{i=1}^{\infty} x_i \varepsilon^i, \quad y = \sum_{i=1}^{\infty} y_i \varepsilon^i, \quad w = \sum_{i=1}^{\infty} w_i \varepsilon^i, \quad (34)$$

where each x_i, y_i, w_i is separately periodic for $|\varepsilon|$ sufficiently small.

Now let us suppose that α is an integer or zero. Then the determinant (33) is zero, and in order to prove the existence of periodic solutions of (16) we require the explicit form of the quadratic terms of the first two equations of (25). Let us denote the linear and quadratic terms of these two equations by x_1, y_1 and x_2, y_2 respectively. Then the values of x_1 and y_1, subject to the initial conditions (24), are

$$\left.\begin{aligned}
x_1 &= \left(\frac{a_1}{2} - \frac{a_2}{\Delta}\right) e^{a\sqrt{-1}\tau}\xi_1 + \left(\frac{a_1}{2} + \frac{a_2}{\Delta}\right) e^{-a\sqrt{-1}\tau}\xi_2,\\
y_1 &= \left(\frac{a_3}{2} - \frac{a_4}{\Delta}\right) e^{a\sqrt{-1}\tau}\xi_1 + \left(\frac{a_3}{2} + \frac{a_4}{\Delta}\right) e^{-a\sqrt{-1}\tau}\xi_2.
\end{aligned}\right\} \quad (35)$$

The differential equations from which x_2 and y_2 are obtained, are the same as (17) except that the right members are not zero. If the right members are denoted by $X^{(2)}$ and $Y^{(2)}$ respectively, then

$$X^{(2)} = x_1 w_1 R_0 + \varepsilon x_1 R_1 + \varepsilon y_1 R_2 + \varepsilon w_1 R_3 + \varepsilon^2 R_4,$$
$$Y^{(2)} = y_1 w_1 R_0 + \varepsilon x_1 R_2 + \varepsilon y_1 R_5 + \varepsilon w_1 R_6 + \varepsilon^2 R_7,$$

where R_0 is a power series in μ with constant coefficients, and $R_i\,(i=1,\ldots,7)$ are power series in μ in which the coefficients are sums of cosines of even multiples of τ. The complementary functions of the differential equations defining x_2, y_2 are the same as (18); that is,

$$\left.\begin{aligned} x_2 &= a_1^{(2)} e^{a\sqrt{-1}\,\tau}\xi_1 + a_2^{(2)} e^{-a\sqrt{-1}\,\tau}\xi_2, \\ y_2 &= a_3^{(2)} e^{a\sqrt{-1}\,\tau}\xi_1 + a_4^{(2)} e^{-a\sqrt{-1}\,\tau}\xi_2, \end{aligned}\right\} \tag{36}$$

where $a_i^{(2)}\,(i=1,\ldots,4)$ are the arbitrary constants. Now regarding these constants as variables, according to the method of the variation of parameters, we have

$$\left.\begin{aligned} a_1^{(2)} &= -\frac{1}{\Delta} e^{-a\sqrt{-1}\,\tau}\xi_2 X^{(2)}, & a^{(2)} &= \frac{1}{\Delta} e^{a\sqrt{-1}\,\tau}\xi_1 X^{(2)}, \\ a_3^{(2)} &= -\frac{1}{\Delta} e^{-a\sqrt{-1}\,\tau}\xi_2 Y^{(2)}, & a_4^{(2)} &= \frac{1}{\Delta} e^{a\sqrt{-1}\,\tau}\xi_1 Y^{(2)}, \end{aligned}\right\} \tag{37}$$

where Δ is the determinant of the fundamental set of solutions (18), and is therefore different from zero. Since $X^{(2)}$, $Y^{(2)}$ contain terms in $e^{\pm a\sqrt{-1}\,\tau}$ multiplied by power series in μ in which the coefficients are sums of cosines of even multiples of τ, the integration of the equations (37) will yield non-periodic terms. We shall be concerned with the explicit form of only the non-periodic terms arising from (37). When the values of $a_i^{(2)}\,(i=1,\ldots,4)$ are obtained from (37) and substitued in (36), we have

$$\left.\begin{aligned} x_2 = {} & \tau\,(\varepsilon p^{(1)} + a_5 p^{(2)})\Big[\Big(\frac{a_1}{2} - \frac{a_2}{\Delta}\Big) e^{a\sqrt{-1}\,\tau}\xi_1 - \Big(\frac{a_1}{2} + \frac{a_2}{\Delta}\Big) e^{-a\sqrt{-1}\,\tau}\xi_2\Big] \\ & + \tau\varepsilon p^{(3)}\Big[\Big(\frac{a_3}{2} - \frac{a_4}{\Delta}\Big) e^{a\sqrt{-1}\,\tau}\xi_1 - \Big(\frac{a_3}{2} + \frac{a_4}{\Delta}\Big) e^{-a\sqrt{-1}\,\tau}\xi_2\Big] \\ & + \varepsilon a_5\,[\text{non-periodic terms}] + \text{periodic terms}, \\ y_2 = {} & \tau\varepsilon p^{(3)}\Big[\Big(\frac{a_1}{2} - \frac{a_2}{\Delta}\Big) e^{a\sqrt{-1}\,\tau}\xi_1 - \Big(\frac{a_1}{2} + \frac{a_2}{\Delta}\Big) e^{-a\sqrt{-1}\,\tau}\xi_2\Big] \\ & + \tau\,(\varepsilon p^{(4)} + a_5 p^{(2)})\Big[\Big(\frac{a_3}{2} - \frac{a_4}{\Delta}\Big) e^{a\sqrt{-1}\,\tau}\xi_1 - \Big(\frac{a_3}{2} + \frac{a_4}{\Delta}\Big) e^{-a\sqrt{-1}\,\tau}\xi_2\Big] \\ & + \varepsilon a_5\,[\text{non-periodic terms}] + \text{periodic terms}, \end{aligned}\right\} \tag{38}$$

where $p^{(i)}\,(i=1,\ldots,4)$ are power series in μ with constant coefficients. Hence, the solutions of (16) as power series in ε and $a_i\,(i=1,\ldots,5)$ are

$$\left.\begin{aligned} x &= x_1 + x_2 + \dots, \\ y &= y_1 + y_2 + \dots, \\ w &= \frac{a_5}{\chi(0)}[\chi + A\tau\dot{\phi}] + \varepsilon P_3(a_1, \dots, a_5, \varepsilon;\ \tau), \end{aligned}\right\} \tag{39}$$

where x_1, y_1; x_2, y_2; and P_3 are defined in (35), (38) and (25) respectively.

Now let us impose upon (39) the necessary periodicity conditions of (26). Since x_1, y_1 are periodic when α is an integer, we obtain

$$\left.\begin{aligned} 0 &= -\frac{4\pi a_2}{\Delta}(\varepsilon p^{(1)} + a_5 p^{(2)} - \frac{4\pi a_4}{\Delta}\varepsilon p^{(3)} + \varepsilon a_5 r^{(1)} \\ &\quad + \varepsilon\,[\text{quadratic and higher degree terms in } \varepsilon, a_1, \dots, a_5], \\ 0 &= -\pi\Delta a_1(\varepsilon p^{(1)} + a_5 p^{(2)}) - \pi\Delta a_3 \varepsilon p^{(3)} + \varepsilon a_5 r^{(2)} \\ &\quad + \varepsilon\,[\text{quadratic and higher degree terms in } \varepsilon, a_1, \dots, a_5], \\ 0 &= -\frac{4\pi a_2}{\Delta}\varepsilon p^{(3)} - \frac{4\pi a_4}{\Delta}(\varepsilon p^{(4)} + a_5 p^{(2)}) + \varepsilon a_5 r^{(3)} \\ &\quad + \varepsilon\,[\text{quadratic and higher degree terms in } \varepsilon, a_1, \dots, a_5], \\ 0 &= -\pi\Delta a_1 \varepsilon p^{(3)} - \pi\Delta a_3(\varepsilon p^{(4)} + a_5 p^{(2)}) + \varepsilon a_5 r^{(4)} \\ &\quad + \varepsilon\,[\text{quadratic and higher degree terms in } \varepsilon, a_1, \dots, a_5], \\ 0 &= \frac{2\pi A a_5 \dot{\phi}(2\pi)}{\chi(0)} + \varepsilon\,[\text{terms in } \varepsilon, a_1, \dots, a_5], \end{aligned}\right\} \tag{40}$$

where $r^{(i)}$ $(i = 1, \dots, 4)$ denote power series in μ with constant coefficients. The coefficient of a_5 in the last equation of (40) is different from zero and therefore this equation can be solved uniquely for a_5 as a power series in ε and a_j $(j = 1, \dots, 4)$ of the form

$$a_5 = \varepsilon\, p\,(a_1, \dots, a_4, \varepsilon). \tag{41}$$

When (41) is substituted for a_5 in the first four equations of (40), the factor ε can be divided out and we obtain four equations in ε and a_j $(j = 1, \dots, 4)$ of the form

$$\left.\begin{aligned} 0 &= -\frac{4\pi q^{(1)} a_2}{\Delta} - \frac{4\pi p^{(3)} a_4}{\Delta} + \varepsilon Q_1(\varepsilon, a_1, \dots, a_4), \\ 0 &= -\pi\Delta q^{(1)} a_1 - \pi\Delta p^{(3)} a_3 + \varepsilon Q_2(\varepsilon, a_1, \dots, a_4), \\ 0 &= -\frac{4\pi p^{(3)} a_2}{\Delta} - \frac{4\pi q^{(2)} a_4}{\Delta} + \varepsilon Q_3(\varepsilon, a_1, \dots, a_4), \\ 0 &= -\pi\Delta p^{(3)} a_1 - \pi\Delta q^{(2)} a_3 + \varepsilon Q_4(\varepsilon, a_1, \dots, a_4), \end{aligned}\right\} \tag{42}$$

where $q^{(1)}$, $q^{(2)}$ are power series in μ with constant coefficients, and Q_i $(i = 1, \dots, 4)$ are power series in ε and a_i $(i = 1, \dots, 4)$ in which the coefficients

are power series in μ. The determinant of the coefficients of the linear terms in a_i in (42) is

$$16\,\pi^4\,\{q^{(1)}\,q^{(2)}-(p^{(3)})^2\}^2. \tag{43}$$

This determinant is not zero, in general, but it may be possible to choose such values of the constants c_1, c_2 and f_{ijk} (see equation (2)) that (43) shall vanish. We shall exclude such special values of these constants, if any exist, and therefore (43) is not zero. Hence, (42) can be solved uniquely for $a_j\,(j=1,\ldots,4)$ as power series in ε, vanishing with ε. When these series for a_j are substituted in (41), a_5 is likewise a power series in ε, vanishing with ε. Consequently, when α is an integer or zero, periodic solutions of (16) exist uniquely and are power series in ε of the form

$$x=\sum_{i=1}^{\infty} x^{(i)}\,\varepsilon^i, \qquad y=\sum_{i=1}^{\infty} y^{(i)}\,\varepsilon^i, \qquad w=\sum_{i=1}^{\infty} w^{(i)}\,\varepsilon^i, \tag{44}$$

where each $x^{(i)}$, $y^{(i)}$, $w^{(i)}$ is separately periodic for $|\varepsilon|$ sufficiently small.

Now the solutions (44) include the solutions (34) as the special case when α is zero; and since both the solutions are unique, they are therefore identical. Consequently, in making the practical construction of the periodic solutions of (16), it is not necessary to make a special consideration of the case when α is a real integer.

If the period were chosen to be $2\,\nu\,\pi$, ν an integer, then the proof of the existence of periodic solutions of (16) with the period $2\,\nu\,\pi$ would be identical with the preceding proof except that, in the preceding, $2\,\pi$ would be replaced by $2\,\nu\,\pi$. Then periodic solutions exist having the same form as (34). Since these solutions are unique for every ν, and since the orbits having the period $2\,\nu\,\pi$ include those having the period $2\,\pi$, there are no orbits with the period $2\,\nu\,\pi$ which do not have the period $2\,\pi$ also.

If α is a rational fraction N/n, where N and n are real integers relatively prime, the question might be raised whether orbits exist which have the period $2n\pi$ in τ, and not the period 2π. From the preceding paragraph we conclude that such orbits do not exist.

§ 6. *Direct Construction of the Periodic Solutions.*

Let us substitute (34) in (16) and equate the coefficients of the same powers of ε. Since the results are identities in ε, there is obtained a series of

differential equations from which the coefficients in (34) can be determined. The constants of integration arising at each step are to be determined so that the solutions shall be periodic and satisfy the initial condition $w=0$, from which it follows that

$$w_j(0)=0, \qquad (j=1,\ldots,\infty). \tag{45}$$

The differential equations for the terms in ε are

$$\left.\begin{aligned}\ddot{x}_1+[a^2+\theta_1^{(1)}\mu+\theta_2^{(1)}\mu^2+\ldots]\,x_1&=R_1^{(1)},\\ \ddot{y}_1+[a^2+\theta_1^{(1)}\mu+\theta_2^{(1)}\mu^2+\ldots]\,y_1&=R_2^{(1)},\\ \ddot{w}_1+[b^2+\theta_1^{(2)}\mu+\theta_2^{(2)}\mu^2+\ldots]\,w_1&=R_3^{(1)},\end{aligned}\right\} \tag{46}$$

where the $R_i^{(1)}\,(i=1,2,3)$ are power series in μ with sums of cosines of even multiples of τ in the coefficients. The complementary functions of (46) are the same as (18) and (23); that is,

$$\left.\begin{aligned}x_1&=a_1^{(1)}e^{a\sqrt{-1}\,\tau}\xi_1+a_2^{(1)}e^{-a\sqrt{-1}\,\tau}\xi_2,\\ y_1&=a_3^{(1)}e^{a\sqrt{-1}\,\tau}\xi_1+a_4^{(1)}e^{-a\sqrt{-1}\,\tau}\xi_2,\\ w_1&=a_5^{(1)}\phi+a_6^{(1)}[\chi+A\tau\phi],\end{aligned}\right\} \tag{47}$$

where the $a_i^{(1)}\,(i=1,\ldots,6)$ are the constants of integration. On using the method of the variation of parameters, we have

$$\left.\begin{aligned}\Delta\dot{a}_1^{(1)}&=-e^{-a\sqrt{-1}\,\tau}\xi_2R_1^{(1)}, & \Delta\dot{a}_2^{(1)}&=e^{a\sqrt{-1}\,\tau}\xi_1R_1^{(1)},\\ \Delta\dot{a}_3^{(1)}&=-e^{-a\sqrt{-1}\,\tau}\xi_2R_2^{(1)}, & \Delta\dot{a}_4^{(1)}&=e^{a\sqrt{-1}\,\tau}\xi_1R_2^{(1)},\\ D\dot{a}_5^{(1)}&=-[\chi+A\tau\phi]R_3^{(1)}, & D\dot{a}_6^{(1)}&=\phi R_3^{(1)},\end{aligned}\right\} \tag{48}$$

where Δ and D are the determinants of fundamental sets of solutions (18) and (23) respectively and therefore are not zero.

When equations (48) are integrated and the resulting values for $a_i^{(1)}$ are substituted in (47), we obtain the general solutions of (36) which are

$$\left.\begin{aligned}x_1&=A_1^{(1)}e^{a\sqrt{-1}\,\tau}\xi_1+A_2^{(1)}e^{-a\sqrt{-1}\,\tau}\xi_2+C_1^{(1)}(\tau),\\ y_1&=A_3^{(1)}e^{a\sqrt{-1}\,\tau}\xi_1+A_4^{(1)}e^{-a\sqrt{-1}\,\tau}\xi_2+C_2^{(1)}(\tau),\\ w_1&=A_5^{(1)}\phi+A_6^{(1)}[\chi+A\tau\phi]-p_1\tau\phi+C_3^{(1)}(\tau),\end{aligned}\right\} \tag{49}$$

where $A_i^{(1)}\,(i=1,\ldots,6)$ are the constants of integration; $C_i^{(1)}(\tau)\,(i=1,2,3)$ are power series in μ with sums of cosines of even multiples of τ in the coefficients; and p_1 is a power series in μ with constant coefficients.

Since we have shown that the periodic solutions of (16) have the same form whether α is an integer or not, the constants $A_i^{(1)}\,(i=1,\ldots,4)$ must be zero in order that the first two equations of (49) shall be periodic when α is not an integer. If these constants are not put equal to zero at this step, a consideration of the terms in ε^2 will show that they must be zero in order that x_2 and y_2 shall be periodic.

In order that the last equation of (49) shall be periodic, the constant $A_6^{(1)}$ must have the value

$$A_6^{(1)}=\frac{p_1}{A}=\frac{1}{\mu}q_1,$$

where q_1 is a power series in μ with constant coefficients. From the condition (45) it follows that $A_5^{(1)}=0$. When these values of $A_i^{(1)}\,(i=1,\ldots,6)$ are substituted in (49), the solutions of (46) become

$$x_1=C_1^{(1)}(\tau),\quad y_1=C_2^{(1)}(\tau),\quad w_1=\frac{1}{\mu}q_1\chi+C_3^{(1)}(\tau)=\frac{1}{\mu}\overline{C}_1(\tau),\tag{50}$$

where $\overline{C}_1(\tau)$ has the same form as $C_i^{(1)}\,(i=1,2,3)$.

It is easy to show that all the succeeding steps of the integration are entirely similar. The differential equations for the coefficients of ε^n are

$$\left.\begin{aligned}\ddot{x}_n+[a^2+\theta_1^{(1)}\mu+\theta_2^{(1)}\mu^2+\ldots]\,x_n&=\frac{1}{\mu^{n-1}}R_1^{(n)},\\ \ddot{y}_n+[a^2+\theta_1^{(1)}\mu+\theta_2^{(1)}\mu^2+\ldots]\,y_n&=\frac{1}{\mu^{n-1}}R_2^{(n)},\\ \ddot{w}_n+[b^2+\theta_1^{(2)}\mu+\theta_2^{(2)}\mu^2+\ldots]\,w_n&=\frac{1}{\mu^{n-1}}R_3^{(n)},\end{aligned}\right\}\tag{51}$$

where $R_i^{(n)}\,(i=1,2,3)$ are similar in form to $R_i^{(1)}\,(i=1,2,3)$. Then on forming the equations analogous to (47) and (48), we obtain the general solutions of (51), viz.,

$$\left.\begin{aligned}x_n&=A_1^{(n)}e^{a\sqrt{-1}\,\tau}\,\xi_1+A_2^{(n)}e^{-a\sqrt{-1}\,\tau}\,\xi_2+\frac{1}{\mu^{n-1}}C_1^{(n)}(\tau),\\ y_n&=A_3^{(n)}e^{a\sqrt{-1}\,\tau}\,\xi_1+A_4^{(n)}e^{-a\sqrt{-1}\,\tau}\,\xi_2+\frac{1}{\mu^{n-1}}C_2^{(n)}(\tau),\\ w_n&=A_5^{(n)}\phi+A_6^{(n)}[\chi+A\tau\phi]-\frac{1}{\mu^{n-1}}p_n\tau\phi+\frac{1}{\mu^{n-1}}C_3^{(n)}(\tau),\end{aligned}\right\}\tag{52}$$

where $A_i^{(n)}\,(i=1,\ldots,6)$ are the constants of integration; $C_i^{(n)}(\tau)\,(i=1,2,3)$ are periodic functions similar in form to $C_i^{(1)}\,(i=1,2,3)$; and p_n is a power series in μ with constant coefficients. In order that the solutions (52) shall

satisfy the periodicity and the initial conditions, the constants of integration must have the values

$$A_i^{(n)} = 0 \ (i = 1, \ldots, 5), \quad A_6^{(n)} = \frac{1}{\mu^{n-1}} \frac{p_n}{A} = \frac{1}{\mu^n} q_n,$$

where q_n is a power series in μ with constant coefficients. Hence, the desired solutions of (51) are

$$\left.\begin{aligned} x_n &= \frac{1}{\mu^{n-1}} C_1^{(n)}(\tau), \\ y_n &= \frac{1}{\mu^{n-1}} C_2^{(n)}(\tau), \\ w_n &= \frac{1}{\mu^n} q_n \chi + \frac{1}{\mu^{n-1}} C_3^{(n)}(\tau) = \frac{1}{\mu^n} \bar{C}_n(\tau), \end{aligned}\right\} \tag{53}$$

where $\bar{C}_n(\tau)$ is similar in form to $\bar{C}_1(\tau)$. Thus the general step of the integration is entirely similar to the first step.

When we are dealing with Cases II and III of the solutions of the first two equations of (17), the method of proving the existence and of making the construction of the periodic solutions of (16) is similar to the preceding. In the other two cases, as in Case I, the solutions of (16) are power series in ε. In Case II the coefficients of the various powers of ε are power series in μ similar to those obtained in (53), but they contain additional terms in $\cos 2(aj + k)\tau$, j and k integers. In Case III the coefficients of the various powers of ε are power series in $\sqrt{\mu}$ with coefficients similar to those in (53).

§ 7. *The Character of the Surface.*

The equation of the surface is general except that $|\varepsilon|$ must be taken small in order to insure the convergence of certain series appearing in the preceding. Suppose the periodic orbits are desired on a given surface S. Because $|\varepsilon|$ is small, it may not be possible to choose such values of ε and f_{ijk} in (2) that (1) shall represent S. Let us suppose that certain values of f_{ijk} are taken in (1) and that $\bar{\varepsilon}$ is the largest value $|\varepsilon|$ may take, and let us denote the resulting equation of the surface by S_1. Then by the preceding method we can determine periodic orbits on the surface

$$S_2 \equiv S_1 + 2\varepsilon_1 f_1(x, y, z) = 0,$$

where ε_1 is a new parameter and $f_1(x, y, z)$ has the same form as $f(x, y, z)$. By repeating the same process over and over again, we can determine periodic orbits on a sequence of surfaces

$$S_{k+1} \equiv S_k + 2\,\varepsilon_k f_k(x, y, z) = 0, \qquad (k = 1, 2, 3, \ldots .),$$

where $f_k(x, y, z)$ has the same form as (2) and ε_k is a parameter, *provided, of course, that the solutions obtained do not pass through any singularities.* Thus, in general, the given surface S can be approached by the sequence of surfaces S_{k+1}, and the periodic orbits described on it can be obtained as power series in $\mu, \varepsilon, \varepsilon_1, \ldots, \varepsilon_k, \ldots .$

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